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Vibration Mode Localization in Two-Dimensional Systems

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The method of regular perturbation is applied to study vibration mode localization in randomly disordered weakly coupled two-dimensional cantilever–spring arrays. Localization factors, which characterize the average exponential rates of decay or growth of the amplitudes of vibration, are defined in terms of the angles of orientation. First-order approximate results of the localization factor are obtained using a combined analytical-numerical approach. The localization factors are symmetric about the cantilever and the horizontal and vertical axes passing through the cantilever at which vibration is originated. For the systems under consideration, the direction in which vibration is originated corresponds to the smallest localization factor; whereas the diagonal directions correspond to the largest rate of decay or growth of the amplitudes of vibration. When plotted in the logarithmic scale, the vibration modes are of a hill shape with the amplitudes of vibration decaying linearly away from the cantilever at which vibration is originated.

Nomenclature

$\boldsymbol{A}$	= diagonal matrix obtained from the main diagonal of matrix $\bar{\boldsymbol{A}}$
$\bar{\boldsymbol{A}}$	= $N \times N$ matrix, where N is equal to $2N_H N_V$
$d^l_{I,J}$, $d^r_{I,J}$	= lengths of the left-slanting and right-slanting diagonals of panel I, J
h_I	= height of row I
int(x)	= integral part of real number x
$\boldsymbol{K}$	= stiffness matrix
$\hat{K}^{\text{direction}}_{I,J}$	= $K^{\text{direction}}_{I,J}/K^x$, where direction is equal to x, y, h, v, dl, dr, nondimensional spring stiffnesses
$K^{dl}_{I,J}$, $K^{dr}_{I,J}$	= stiffnesses of the left-slanting and right-slanting springs in panel I, J
$K^h_{I,J}$	= stiffness of the horizontal spring on line I and in column J
$K^v_{I,J}$	= stiffness of the vertical spring in row I and on line J
K^x	= average value of the bending stiffness $K^x_{I,J}$
$K^x_{I,J}$, $K^y_{I,J}$	= bending stiffnesses of the (I, J)th cantilever in the x and y directions
k^h, h^v, h^d	= nondimensional spring stiffnesses defined by Eq. (A11)
k_j, k_{j+1}	= nondimensional spring stiffnesses defined by Eq. (A14)
k^x_j	= nondimensional bending stiffness of the cantilever (I, J) in the x direction, $\hat{K}^x_{I,J}$
k^y_{j+1}	= nondimensional bending stiffness of the cantilever (I, J) in the y direction, $\hat{K}^y_{I,J}$
L	= length of the cantilevers
l_J	= width of column J
$\boldsymbol{M}$	= mass matrix
m	= average value of the masses $m_{I,J}$
$m_{I,J}$	= lumped mass at the tip of cantilever (I, J)
$\hat{m}_{I,J}$	= $m_{I,J}/m$
N_H	= number of columns of cantilevers in the array
N_V	= number of rows of cantilevers in the array
u_j, u_{j+1}	= nondimensional displacements in the x and y directions
$\boldsymbol{u}_p$	= vector with all elements being zero except the pth element, which is 1
$\bar{\boldsymbol{u}}_p$	= mode in which vibration is originated at the pth global coordinate
$x_{I,J}$, $y_{I,J}$	= horizontal and vertical displacement components of node (I, J)
$\delta\boldsymbol{A}$	= matrix obtained from the off-diagonal terms of $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{A}} - \boldsymbol{A}$
$\delta^i\bar{\boldsymbol{u}}_p$	= the ith-order perturbation of $\bar{\boldsymbol{u}}_p$, $\delta^0\boldsymbol{u}_p$ is equal to $\boldsymbol{u}_p$
$\delta^i\nu_p$	= the ith-order perturbation of $\bar{\nu}_p$, $\delta^0\nu_p$ is equal to ν_p
δ_X	= coefficient of variation of the random variable X
ε^M_{pq}	= amplitude of vibration of the qth global coordinate in the Mth-order perturbation for the mode in which vibration is originated at the pth global coordinate; p, q odd for x direction and even for y direction
θ	= angle of orientation defined by Eq. (17) and Fig. 5
λ_θ	= localization factor in direction θ, Eq. (16)
μ_X	= mean value of the random variable X
ν	= nondimensional natural frequency, ω^2/ω_0^2
ν_p	= unperturbed value of $\bar{\nu}_p$
$\bar{\nu}_p$	= eigenvalue for the mode in which vibration is originated at the pth global coordinate
σ_X	= standard deviation of the random variable X
ω	= natural frequency of vibration
ω_0^2	= K^x/m
$\Vert\,\Vert$	= suitable vector norm

Superscripts

h, v, d	= horizontal, vertical, and diagonal directions, respectively

I. Introduction

In the companion paper,[1] the method of regular perturbation was extended to include perturbation terms of infinite orders and was then applied to study vibration mode localization in randomly disordered weakly coupled one-dimensional cantilever–spring chains. First-order approximate results for the localization factors were obtained using the method of perturbation, which characterize the average exponential rates of decay or growth of the amplitudes of cantilevers.

In linear vibration problems, the natural frequencies and the corresponding normal modes of vibration are given by the eigenvalues and the corresponding eigenvectors of a linear eigenvalue problem $(\bar{\boldsymbol{A}} - \bar{\nu}\boldsymbol{I})\bar{\boldsymbol{u}} = \boldsymbol{0}$. For multicoupled one-dimensional systems, matrix $\bar{\boldsymbol{A}}$ is of a block tridiagonal form; the localization factors may be determined using the method of perturbation, the method of transfer matrix, and the method of Green's function. A review of vibration localization in one-dimensional disordered systems was presented in Ref. 1 and will not be repeated here.

For two-dimensional disordered systems, matrix $\bar{\boldsymbol{A}}$ is no longer a block tridiagonal matrix but a general sparse matrix; the methods of transfer matrix and Green's function in their current formulations are not applicable to determine the localization factors of vibration

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modes. Studies on the localization phenomenon in structural dynamics have been restricted to one-dimensional systems. Because of the degree of difficulty and the amount of computation involved in studying higher-dimensional systems, no research work on localization in these systems has been published in the context of structural dynamics. Employing the method of regular perturbation presented in Ref. 1, vibration mode localization in randomly disordered weakly coupled two-dimensional cantilever–spring arrays is investigated.

II. Perturbation Formulation for Vibration Mode Localization in Weakly Coupled Two-Dimensional Cantilever–Spring Arrays

Consider the free vibration of a randomly disordered two-dimensional cantilever–spring array with weak coupling, as shown in Fig. 1. As shown in Fig. 2, node (I, J) indicates the position at the intersection of the Ith row and the Jth column of the cantilevers. The cantilever at node (I, J) is coupled with the eight adjacent cantilevers by springs in the horizontal, vertical, and diagonal directions. The relationship between the nodal coordinate (I, J) and the global coordinate j is given by Eqs. (A9). As mentioned in the Appendix, it may be assumed that j is an odd number so that it corresponds to the x direction and $j+1$ to the y direction. The nondimensional displacement vector $\bar{\boldsymbol{u}}$ may be written as

$$\bar{\boldsymbol{u}} = \left\{\boldsymbol{v}_1^T, \boldsymbol{v}_3^T, \ldots, \boldsymbol{v}_{2N_H N_V - 1}^T\right\}^T, \qquad \boldsymbol{v}_j^T = \left\{u_j, u_{j+1}\right\}, \qquad j \text{ odd}$$

The nondimensional natural frequencies and the corresponding normal modes of vibration of the two-dimensional cantilever–spring array are the eigenvalues and the corresponding eigenvectors of the eigenvalue problem

$$(\bar{\boldsymbol{A}} - \bar{\nu}\boldsymbol{I})\bar{\boldsymbol{u}} = \boldsymbol{0} \tag{1}$$

where the nondimensional displacement vector $\bar{\boldsymbol{u}}$ is given in Eq. (A10) and matrix $\bar{\boldsymbol{A}}$ is a sparse matrix whose elements are derived in the Appendix. For simplicity of presentation, it is assumed that there is no disorder in the stiffnesses of the linear springs; hence, the nondimensional stiffnesses of the horizontal, vertical, and diagonal springs are k^h, k^v, and k^d, respectively. The nonzero elements in the jth row (corresponding to the x direction) and the $(j+1)$th row (corresponding to the y direction) of matrix $\bar{\boldsymbol{A}}$ are given by Eqs. (A12) and (A13).

The dimension of the displacement vector is $2N_H N_V$, which is very large for large values of N_H and N_V. The large dimensions of the system render numerical methods prohibitive, even for algorithms that take into account the special sparsity of matrix $\bar{\boldsymbol{A}}$, such as the Lanczos algorithm. This is probably the reason that no research work on vibration mode localization in higher-dimensional systems has been published in the context of structural dynamics.

For weak coupling, i.e., when the values of k^h, k^v, and k^d are much smaller than those of k_j and k_{j+1}, matrix $\bar{\boldsymbol{A}}$ may be written as

$$\bar{\boldsymbol{A}} = \boldsymbol{A} + \delta\boldsymbol{A} \tag{2}$$

where

$$\boldsymbol{A} = \operatorname{diag}\left\{k_1, k_2; k_3, k_4; \ldots; k_{2N_H N_V - 1}, k_{2N_H N_V}\right\} \tag{3}$$

The method of regular perturbation presented in Ref. 1 is now applied to solve the eigenvalue problem (1). For the mode in which vibration is originated at cantilever (I_0, J_0) in the x direction or in the jth global coordinate (j odd), the unperturbed eigenvalue and eigenvector are $\nu_j = k_j$ and $\boldsymbol{u}_j$, respectively. The Mth-order perturbation ($M \geq 1$) of the eigenvalue $\bar{\nu}_j$ is given by

$$\begin{aligned}
\delta^M \nu_j &= \sum_{k=1}^{2N_H N_V} \varepsilon_{j,k}^{M-1}\left(\boldsymbol{u}_j^T \delta\boldsymbol{A}\boldsymbol{u}_k\right) - \sum_{m=1}^{M-1} \delta^{M-m}\nu_j \varepsilon_{j,j}^m \\
&= k^d\left(-\varepsilon_{j,j-2N_H-2}^{M-1} + \varepsilon_{j,j-2N_H-1}^{M-1} - \varepsilon_{j,j-2N_H+2}^{M-1}\right. \\
&\quad - \varepsilon_{j,j-2N_H+3}^{M-1} - \varepsilon_{j,j+2N_H-2}^{M-1} - \varepsilon_{j,j+2N_H-1}^{M-1} \\
&\quad \left. - \varepsilon_{j,j+2N_H+2}^{M-1} + \varepsilon_{j,j+2N_H+3}^{M-1}\right) - k^h\left(\varepsilon_{j,j-2}^{M-1} + \varepsilon_{j,j+2}^{M-1}\right) \\
&\quad - \sum_{m=1}^{M-1} \delta^{M-m}\nu_j \varepsilon_{j,j}^m
\end{aligned} \tag{4}$$

[from Eq. (17) in Ref. 1]. The coefficient $\varepsilon_{j,i}^M$ of the Mth-order perturbation ($M \geq 1$), which is the amplitude of vibration of the ith global coordinate (i odd, in the x direction), is given by

$$\begin{aligned}
\varepsilon_{j,i}^M &= \frac{1}{\nu_j - \nu_i}\left[\sum_{k=1}^{2N_H N_V} \varepsilon_{j,k}^{M-1}\left(\boldsymbol{u}_i^T \delta\boldsymbol{A}\boldsymbol{u}_k\right) - \sum_{m=1}^{M-1} \delta^{M-m}\nu_j \varepsilon_{j,i}^m\right] \\
&= \frac{1}{\nu_j - \nu_i}\left[k^d\left(-\varepsilon_{j,i-2N_H-2}^{M-1} + \varepsilon_{j,i-2N_H-1}^{M-1} - \varepsilon_{j,i-2N_H+2}^{M-1}\right.\right. \\
&\quad - \varepsilon_{j,i-2N_H+3}^{M-1} - \varepsilon_{j,i+2N_H-2}^{M-1} - \varepsilon_{j,i+2N_H-1}^{M-1} - \varepsilon_{j,i+2N_H+2}^{M-1} \\
&\quad \left.\left. + \varepsilon_{j,i+2N_H+3}^{M-1}\right) - k^h\left(\varepsilon_{j,i-2}^{M-1} + \varepsilon_{j,i+2}^{M-1}\right) - \sum_{m=1}^{M-1} \delta^{M-m}\nu_j \varepsilon_{j,i}^m\right]
\end{aligned} \tag{5}$$

[from Eq. (16) in Ref. 1], whereas the coefficient $\varepsilon_{j,i+1}^M$ of the Mth-order perturbation ($M \geq 1$), which is the amplitude of vibration of the $(i+1)$th global coordinate ($i+1$ even, in the y direction), is given by

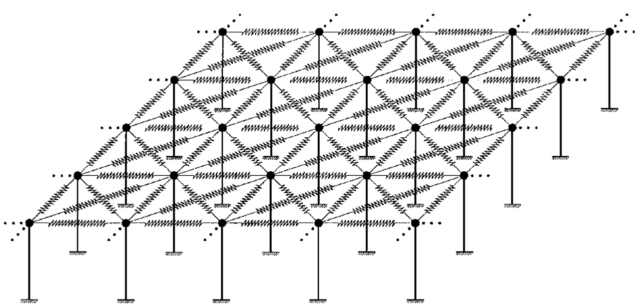
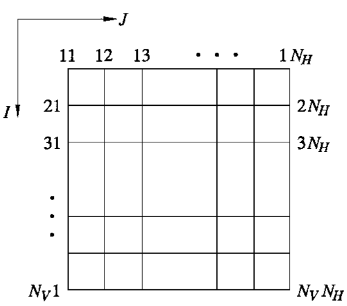
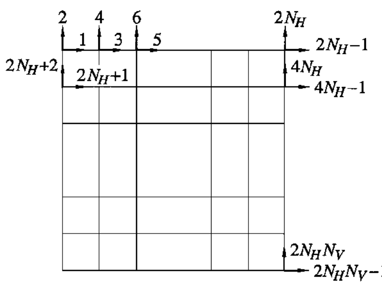


Fig. 1 Two-dimensional cantilever–spring array.



Nodal coordinates (*I*, *J*)



Global coordinates (*j*)

Fig. 2 Nodal coordinates and global coordinates.

$$
\begin{aligned}
\varepsilon^M_{j,i+1} &= \frac{1}{\nu_j - \nu_{i+1}}\left[\sum_{k=1}^{2N_HN_V}\varepsilon^{M-1}_{j,k}(\boldsymbol{u}^T_{i+1}\delta \boldsymbol{A}\boldsymbol{u}_k) - \sum_{m=1}^{M-1}\delta^{M-m}\nu_j\varepsilon^m_{j,i+1}\right] \\
&= \frac{1}{\nu_j - \nu_{i+1}}\Big[k^d\Big(+\varepsilon^{M-1}_{j,i-2N_H-2} - \varepsilon^{M-1}_{j,i-2N_H-1} - \varepsilon^{M-1}_{j,i-2N_H+2} \\
&\quad - \varepsilon^{M-1}_{j,i-2N_H+3} - \varepsilon^{M-1}_{j,i+2N_H-2} - \varepsilon^{M-1}_{j,i+2N_H-1} + \varepsilon^{M-1}_{j,i+2N_H+2} \\
&\quad - \varepsilon^{M-1}_{j,i+2N_H+3}\Big) - k^v\Big(\varepsilon^{M-1}_{j,i-2N_H+1} + \varepsilon^{M-1}_{j,i+2N_H+1}\Big) \\
&\quad - \sum_{m=1}^{M-1}\delta^{M-m}\nu_j\varepsilon^m_{j,i+1}\Big]
\end{aligned}
\tag{6}
$$

Similarly, for the vibration mode in which vibration is originated at cantilever (I_0, J_0) in the y direction or in the $(j+1)$th global coordinate ($j+1$ even), the unperturbed eigenvalue and eigenvector are $\nu_{j+1} = k_{j+1}$ and $\boldsymbol{u}_{j+1}$, respectively. Hence,

$$
\begin{aligned}
\delta^M\nu_{j+1} &= k^d\Big(+\varepsilon^{M-1}_{j+1,j-2N_H-2} - \varepsilon^{M-1}_{j+1,j-2N_H-1} \\
&\quad - \varepsilon^{M-1}_{j+1,j-2N_H+2} - \varepsilon^{M-1}_{j+1,j-2N_H+3} - \varepsilon^{M-1}_{j+1,j+2N_H-2} \\
&\quad - \varepsilon^{M-1}_{j+1,j+2N_H-1} + \varepsilon^{M-1}_{j+1,j+2N_H+2} - \varepsilon^{M-1}_{j+1,j+2N_H+3}\Big) \\
&\quad - k^v\Big(\varepsilon^{M-1}_{j+1,j-2N_H+1} + \varepsilon^{M-1}_{j+1,j+2N_H+1}\Big) \\
&\quad - \sum_{m=1}^{M-1}\delta^{M-m}\nu_{j+1}\varepsilon^m_{j+1,j+1}
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\varepsilon^M_{j+1,i} &= \frac{1}{\nu_{j+1} - \nu_i}\Big[k^d\Big(-\varepsilon^{M-1}_{j+1,i-2N_H-2} + \varepsilon^{M-1}_{j+1,i-2N_H-1} \\
&\quad - \varepsilon^{M-1}_{j+1,i-2N_H+2} - \varepsilon^{M-1}_{j+1,i-2N_H+3} - \varepsilon^{M-1}_{j+1,i+2N_H-2} \\
&\quad - \varepsilon^{M-1}_{j+1,i+2N_H-1} - \varepsilon^{M-1}_{j+1,i+2N_H+2} + \varepsilon^{M-1}_{j+1,i+2N_H+3}\Big) \\
&\quad - k^h(\varepsilon^{M-1}_{j+1,i-2} + \varepsilon^{M-1}_{j+1,i+2}) - \sum_{m=1}^{M-1}\delta^{M-m}\nu_{j+1}\varepsilon^m_{j+1,i}\Big]
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\varepsilon^M_{j+1,i+1} &= \frac{1}{\nu_{j+1} - \nu_{i+1}}\Big[k^d\Big(+\varepsilon^{M-1}_{j+1,i-2N_H-2} - \varepsilon^{M-1}_{j+1,i-2N_H-1} \\
&\quad - \varepsilon^{M-1}_{j+1,i-2N_H+2} - \varepsilon^{M-1}_{j+1,i-2N_H+3} - \varepsilon^{M-1}_{j+1,i+2N_H-2} \\
&\quad - \varepsilon^{M-1}_{j+1,i+2N_H-1} + \varepsilon^{M-1}_{j+1,i+2N_H+2} - \varepsilon^{M-1}_{j+1,i+2N_H+3}\Big) \\
&\quad - k^v\Big(\varepsilon^{M-1}_{j+1,i-2N_H+1} + \varepsilon^{M-1}_{j+1,i+2N_H+1}\Big) \\
&\quad - \sum_{m=1}^{M-1}\delta^{M-m}\nu_{j+1}\varepsilon^m_{j+1,i+1}\Big]
\end{aligned}
\tag{9}
$$

For a one-dimensional cantilever–spring chain, each cantilever is coupled with its two adjacent cantilevers by springs, whereas for a two-dimensional cantilever–spring array, each cantilever is coupled with its eight neighboring cantilevers through the horizontal, vertical, and diagonal springs (Fig. 3). As shown in Fig. 4, for the vibration mode in which vibration is originated at the (I_0, J_0)th cantilever, only the (I_0, J_0)th cantilever is vibrating in the zeroth-order perturbation. For the first-order perturbation, the eight neighboring cantilevers on the first layer are brought into motion, whereas for the second-order perturbation, the 16 cantilevers on the second layer are brought into motion due to coupling. In general, for the Mth-order perturbation, $8M$ cantilever on the Mth layer are brought into motion for the first time. For the mth-order perturbation ($m < M$), all cantilevers on and outside the Mth layer are at rest. Therefore, the Mth layer is the farthest layer that vibration can extend in the Mth-order perturbation. Hence, vibration extends or propagates outward a layer for each increment of M, as shown in Fig. 4.

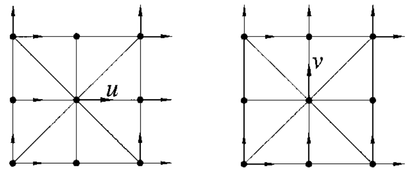


Fig. 3 Coupling of vibration.

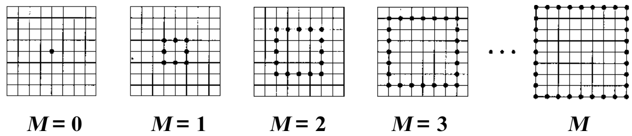


Fig. 4 Extension of vibration of a two-dimensional system.

Therefore, it is clear that the amplitudes of vibration of the cantilevers on the Mth layer in the Mth-order perturbation are determined only by the amplitudes of vibration of the cantilevers on the $(M-1)$th layer in the $(M-1)$th-order perturbation. Let the global coordinate i be on the Mth layer ($M \geq 1$); then

$$
\varepsilon^m_{j,i} = 0, \qquad m = 1, 2, \ldots, M-1 \tag{10}
$$

and Eqs. (5), (6), (8), and (9) become

$$
\begin{aligned}
\varepsilon^M_{j,i} &= \frac{1}{\nu_j - \nu_i}\Big[k^d\Big(-\varepsilon^{M-1}_{j,i-2N_H-2} + \varepsilon^{M-1}_{j,i-2N_H-1} - \varepsilon^{M-1}_{j,i-2N_H+2} \\
&\quad - \varepsilon^{M-1}_{j,i-2N_H+3} - \varepsilon^{M-1}_{j,i+2N_H-2} - \varepsilon^{M-1}_{j,i+2N_H-1} - \varepsilon^{M-1}_{j,i+2N_H+2} \\
&\quad + \varepsilon^{M-1}_{j,i+2N_H+3}\Big) - k^h(\varepsilon^{M-1}_{j,i-2} + \varepsilon^{M-1}_{j,i+2})\Big]
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\varepsilon^M_{j,i+1} &= \frac{1}{\nu_j - \nu_{i+1}}\Big[k^d\Big(+\varepsilon^{M-1}_{j,i-2N_H-2} - \varepsilon^{M-1}_{j,i-2N_H-1} - \varepsilon^{M-1}_{j,i-2N_H+2} \\
&\quad - \varepsilon^{M-1}_{j,i-2N_H+3} - \varepsilon^{M-1}_{j,i+2N_H-2} - \varepsilon^{M-1}_{j,i+2N_H-1} + \varepsilon^{M-1}_{j,i+2N_H+2} \\
&\quad - \varepsilon^{M-1}_{j,i+2N_H+3}\Big) - k^v\Big(\varepsilon^{M-1}_{j,i-2N_H+1} + \varepsilon^{M-1}_{j,i+2N_H+1}\Big)\Big]
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\varepsilon^M_{j+1,i} &= \frac{1}{\nu_{j+1} - \nu_i}\Big[k^d\Big(-\varepsilon^{M-1}_{j+1,i-2N_H-2} + \varepsilon^{M-1}_{j+1,i-2N_H-1} \\
&\quad - \varepsilon^{M-1}_{j+1,i-2N_H+2} - \varepsilon^{M-1}_{j+1,i-2N_H+3} - \varepsilon^{M-1}_{j+1,i+2N_H-2} \\
&\quad - \varepsilon^{M-1}_{j+1,i+2N_H-1} - \varepsilon^{M-1}_{j+1,i+2N_H+2} + \varepsilon^{M-1}_{j+1,i+2N_H+3}\Big) \\
&\quad - k^h(\varepsilon^{M-1}_{j+1,i-2} + \varepsilon^{M-1}_{j+1,i+2})\Big]
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\varepsilon^M_{j+1,i+1} &= \frac{1}{\nu_{j+1} - \nu_{i+1}}\Big[k^d\Big(+\varepsilon^{M-1}_{j+1,i-2N_H-2} - \varepsilon^{M-1}_{j+1,i-2N_H-1} \\
&\quad - \varepsilon^{M-1}_{j+1,i-2N_H+2} - \varepsilon^{M-1}_{j+1,i-2N_H+3} - \varepsilon^{M-1}_{j+1,i+2N_H-2} \\
&\quad - \varepsilon^{M-1}_{j+1,i+2N_H-1} + \varepsilon^{M-1}_{j+1,i+2N_H+2} - \varepsilon^{M-1}_{j+1,i+2N_H+3}\Big) \\
&\quad - k^v\Big(\varepsilon^{M-1}_{j+1,i-2N_H+1} + \varepsilon^{M-1}_{j+1,i+2N_H+1}\Big)\Big]
\end{aligned}
\tag{14}
$$

Iterative equations (11–14) may be applied to determine numerically the amplitudes of vibration of the cantilevers on the Mth layer for M large. These iterative equations have important advantages in numerical computation. The $8M$ cantilevers on the Mth layer are brought into motion for the first time in the Mth-order perturbation. To determine the first-order approximation of the localization factors of a vibration mode, only the amplitudes of vibration of these cantilevers, for M large, are required; the dimension of the displacement vector is only $16M$. Note that the amplitudes of vibration of

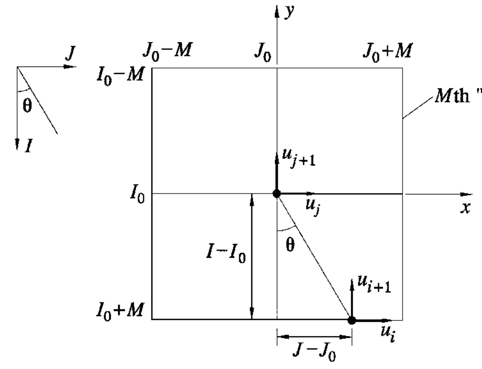


Fig. 5 Displacements of a cantilever on the *M*th layer.

the cantilevers on or inside the $(M-1)$th layer will be modified in the Mth-order perturbation. However, these modifications do not affect the amplitudes of vibration of the cantilevers on the Mth layer in the Mth-order perturbation. Because the amplitudes of vibration of the cantilevers on the Mth layer in the Mth-order perturbation depend only on those of the cantilevers on the $(M-1)$th layer in the $(M-1)$th-order perturbation, only two displacement vectors are needed in numerically solving the iterative equations (11–14). The requirement on computer memory and storage, thus, is reduced dramatically.

Analogous to a one-dimensional randomly disordered system, the localization factors of a two-dimensional randomly disordered system characterize the average exponential rates of growth or decay of amplitudes of vibration. For the vibration mode in which vibration is originated at the (I_0, J_0)th cantilever [corresponding to the jth or $(j+1)$th global coordinates], a localization factor is defined as

$$\|v_i\| \sim e^{-\lambda_\theta M}\|v_j\| \tag{15}$$

or

$$\lambda_\theta = -\lim_{M\to\infty}(1/M)\ln\|v_i\| \tag{16}$$

where $v_i = \{u_i, u_{i+1}\}^T$ is the nondimensional amplitude vector of vibration of the (I, J)th cantilever [corresponding to the ith and $(i+1)$th global coordinates] on the Mth layer and θ is the angle as shown in Fig. 5 and is given by

$$\theta = \tan^{-1}[(J-J_0)/(I-I_0)] \tag{17}$$

Because node (I, J) is on the Mth layer, $|I-I_0| = M$ and/or $|J-J_0| = M$. The localization factors for a two-dimensional system are functions of the angle θ; in different directions, amplitudes of vibration decay or grow at a different rate.

Similar to the analysis of one-dimensional randomly disordered systems, when the leading terms of the amplitudes of vibration, i.e., $\varepsilon^M_{j,i}$ and $\varepsilon^M_{j,i+1}$, or $\varepsilon^M_{j+1,i}$ and $\varepsilon^M_{j+1,i+1}$ given by Eqs. (11–14), are used in Eq. (16), first-order approximations of the localization factors are obtained.

For a one-dimensional, randomly disordered cantilever–spring chain, it was seen in Ref. 1 that, for the vibration mode in which vibration is originated at the jth cantilever, the first-order approximation of the localization factor depends only on the specific physical properties of the jth cantilever and the statistical properties of the chain. For a two-dimensional randomly disordered cantilever–spring array, it is reasonable to expect that, for the vibration mode in which vibration is originated at the (I_0, J_0)th cantilever, the localization factors depend on the specific physical properties of the (I_0, J_0)th cantilever and the statistical properties of the array. The localization factors, therefore, are expected to be symmetric about the $I = I_0$ and $J = J_0$ axes and node (I_0, J_0).

For the four cantilevers at the corners of the Mth layer, which correspond to the angles $\theta = 45$, 135, 225, and 315 deg, Eqs. (11–14) may be simplified significantly. Because of symmetry, only the direction $\theta = 45$ deg is considered in the following without loss of generality. It is obvious that, for the vibration mode in which vibration is originated at the cantilever (I_0, J_0), the amplitudes of vibration of the (I_0+M, J_0+M)th cantilever in the Mth-order perturbation depend only on the amplitudes of vibration of the (I_0+M-1, J_0+M-1)th cantilever in the $(M-1)$th-order perturbation. Hence, Eqs. (11–14) become, for $i = j+2M(N_H+1)$,

$$\varepsilon^M_{j,i} = \frac{k^d}{\nu_j-\nu_i}\left(-\varepsilon^{M-1}_{j,i-2N_H-2}+\varepsilon^{M-1}_{j,i-2N_H-1}\right) \tag{18}$$

$$\varepsilon^M_{j,i+1} = \frac{k^d}{\nu_j-\nu_{i+1}}\left(\varepsilon^{M-1}_{j,i-2N_H-2}-\varepsilon^{M-1}_{j,i-2N_H-1}\right) \tag{19}$$

$$\varepsilon^M_{j+1,i} = \frac{k^d}{\nu_{j+1}-\nu_i}\left(-\varepsilon^{M-1}_{j+1,i-2N_H-2}+\varepsilon^{M-1}_{j+1,i-2N_H-1}\right) \tag{20}$$

$$\varepsilon^M_{j+1,i+1} = \frac{k^d}{\nu_{j+1}-\nu_{i+1}}\left(\varepsilon^{M-1}_{j+1,i-2N_H-2}-\varepsilon^{M-1}_{j+1,i-2N_H-1}\right) \tag{21}$$

Equations (18) and (19) and Eqs. (20) and (21) may be rewritten in the matrix form as

$$\begin{Bmatrix}\varepsilon^M_{j,i}\\ \varepsilon^M_{j,i+1}\end{Bmatrix} = \boldsymbol{T}^x_M\begin{Bmatrix}\varepsilon^{M-1}_{j,i-2N_H-2}\\ \varepsilon^{M-1}_{j,i-2N_H-1}\end{Bmatrix} \tag{22}$$

$$\begin{Bmatrix}\varepsilon^M_{j+1,i}\\ \varepsilon^M_{j+1,i+1}\end{Bmatrix} = \boldsymbol{T}^y_M\begin{Bmatrix}\varepsilon^{M-1}_{j+1,i-2N_H-2}\\ \varepsilon^{M-1}_{j+1,i-2N_H-1}\end{Bmatrix} \tag{23}$$

where $\boldsymbol{T}^x_M$ and $\boldsymbol{T}^y_M$ are transfer matrices given by

$$\boldsymbol{T}^x_M = k^d\begin{bmatrix}-\dfrac{1}{\nu_j-\nu_i} & \dfrac{1}{\nu_j-\nu_i}\\ \dfrac{1}{\nu_j-\nu_{i+1}} & -\dfrac{1}{\nu_j-\nu_{i+1}}\end{bmatrix}$$

$$\boldsymbol{T}^y_M = k^d\begin{bmatrix}-\dfrac{1}{\nu_{j+1}-\nu_i} & \dfrac{1}{\nu_{j+1}-\nu_i}\\ \dfrac{1}{\nu_{j+1}-\nu_{i+1}} & -\dfrac{1}{\nu_{j+1}-\nu_{i+1}}\end{bmatrix}$$

$$i = j+2M(N_H+1)$$

Hence,

$$\begin{Bmatrix}\varepsilon^M_{j,i}\\ \varepsilon^M_{j,i+1}\end{Bmatrix} = \boldsymbol{T}^x_M\boldsymbol{T}^x_{M-1}\cdots\boldsymbol{T}^x_1\begin{Bmatrix}\varepsilon^0_{j,j}\\ \varepsilon^0_{j,j+1}\end{Bmatrix} \tag{24}$$

$$\begin{Bmatrix}\varepsilon^M_{j+1,i}\\ \varepsilon^M_{j+1,i+1}\end{Bmatrix} = \boldsymbol{T}^y_M\boldsymbol{T}^y_{M-1}\cdots\boldsymbol{T}^y_1\begin{Bmatrix}\varepsilon^0_{j+1,j}\\ \varepsilon^0_{j+1,j+1}\end{Bmatrix} \tag{25}$$

Therefore, in the direction of $\theta = 45$ deg, changes of amplitudes of vibration are expressed in terms of products of transfer matrices of dimension 2×2, which implies that the localization behavior of the two-dimensional randomly disordered system in the direction of $\theta = 45$ deg is similar to that of a one-dimensional system. The first-order approximate localization factors, from Eq. (24) for vibration mode in which vibration is originated in the jth global coordinate (x direction), are given by

$$\lambda_{45} = -\lim_{M\to\infty}(1/M)\ln\left\|\boldsymbol{T}^x_M\boldsymbol{T}^x_{M-1}\cdots\boldsymbol{T}^x_1\right\| \tag{26}$$

and from Eq. (25) for vibration mode in which vibration is originated in the $(j+1)$th global coordinate (y direction), are given by

$$\lambda_{45} = -\lim_{M\to\infty}(1/M)\ln\left\|\boldsymbol{T}^y_M\boldsymbol{T}^y_{M-1}\cdots\boldsymbol{T}^y_1\right\| \tag{27}$$

III. Numerical Results

For the vibration mode in which vibration is originated in the jth global coordinate, $\varepsilon^0_{j,j} = 1$ and $\varepsilon^0_{j,j+1} = 0$. Equations (11) and (12) are applied iteratively to determine the amplitudes of vibration of the cantilevers on the Mth layer. For each iteration, two independent uniformly distributed random numbers are generated for the

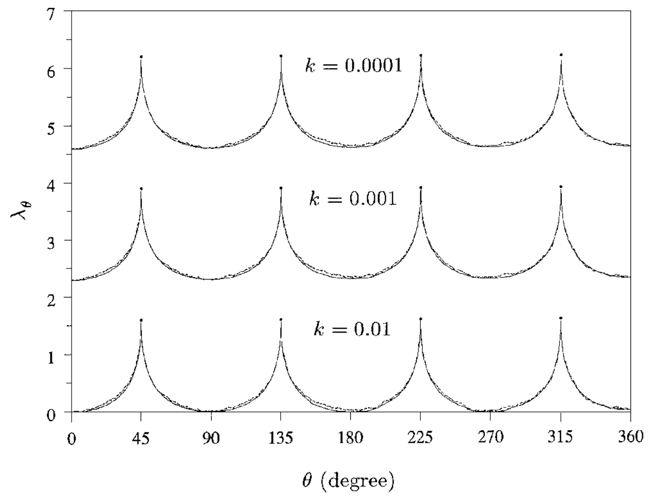


Fig. 6 Localization factors of a two-dimensional system, vibration originated in the *x* direction: $\mu_{kx} = \mu_{ky} = 1$, $\delta_{kx} = \delta_{ky} = 0.1$, $k_j^x = 0.9709$, $k_{j+1}^y = 0.8587$, and $k^h = k^v = k^d = k$; ---, $M = 10^3$; ——, $M = 10^4$; and ● transfer matrix method $M = 10^7$.

nondimensional bending stiffnesses of the cantilever corresponding to the ith global coordinate in the x and y directions, k_i^x and k_i^y, respectively, and $\nu_i = k_i = k_i^x + 2k^h + 4k^d$, $\nu_{i+1} = k_{i+1} = k_i^y + 2k^h + 4k^d$ are calculated. The amplitudes of vibration of the cantilevers on the Mth layer for M large are utilized in Eq. (16) to determine the localization factors λ_θ.

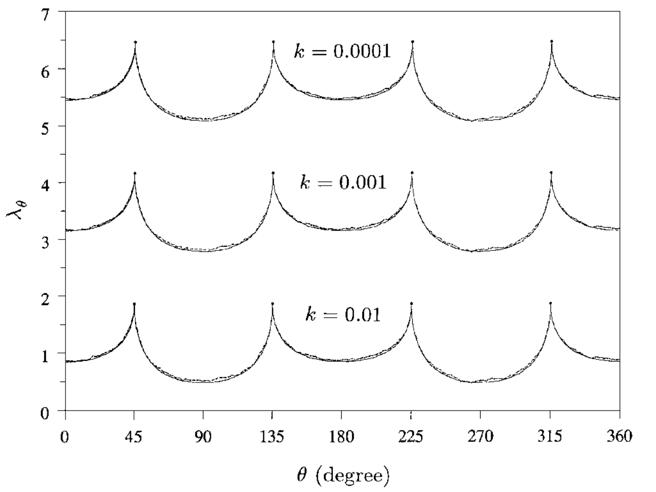
Similarly, for the vibration mode in which vibration is originated in the $(j+1)$th global coordinate, $\varepsilon^0_{j+1,j} = 0$ and $\varepsilon^0_{j+1,j+1} = 1$. Equations (13) and (14) are employed iteratively to determine the amplitudes of vibration of the cantilevers on the Mth layer.

The determination of a first-order approximation of the localization factors requires only the amplitudes of vibration of cantilevers on the Mth layer in the Mth-order perturbation for M large, and all of the cantilevers outside the Mth layer are at rest in the Mth-order perturbation. Therefore, a large two-dimensional cantilever–spring array may be imagined as one with $2M + 1$ rows and $2M + 1$ columns and the cantilever at which vibration is originated being located at node $(M + 1, M + 1)$.

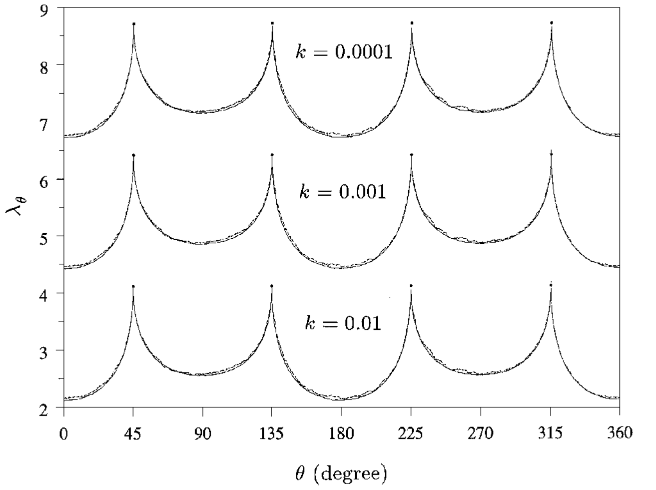
First-order approximate localization factors of a randomly disordered two-dimensional array are shown in Fig. 6 for different values of spring stiffnesses, in which the bending stiffnesses of the cantilevers in the x and y directions are uniformly distributed random numbers with means $\mu_{kx} = \mu_{ky} = 1$ and coefficients of variation $\delta_{kx} = \delta_{ky} = 0.1$, respectively. The bending stiffnesses of the $(M+1, M+1)$th cantilever are randomly generated as $k_j^x = 0.9709$, $k_{j+1}^y = 0.8587$. The localization factors obtained with $M = 10^3$ are shown as dashed lines, whereas those obtained with $M = 10^4$ are shown as solid lines. The good agreement of these two results ascertains that the definition of localization factors given in Eq. (16) is reasonable. It is seen that the localization factor is minimum in the directions of $\theta = 0$, 90, 180, and 270 deg, whereas it attains the maximum value in the directions of $\theta = 45$, 135, 225, and 315 deg. As in one-dimensional cantilever–spring chains, it is also seen that the weaker the coupling, i.e., the smaller the values of the coupling spring stiffness, the larger the localization factors.

The localization factors shown in Fig. 6 are for the vibration mode in which vibration is originated in the x direction. Because the statistical properties of the cantilever–spring array in the x direction are the same as those in the y direction, similar localization behavior is expected for the vibration modes in which vibration is originated in the y direction.

When the bending stiffnesses of the cantilevers in the x and y directions are uniformly distributed random numbers with means $\mu_{kx} = 1$ and $\mu_{ky} = 5$ and coefficients of variation $\delta_{kx} = \delta_{ky} = 0.1$, there are two frequency groups. The first frequency group corresponds to vibration modes in which vibration is originated in the x direction; whereas the second frequency group corresponds to vibration modes in which vibration is originated in the y direction. The



a) Vibration originated in the *x* direction



b) Vibration originated in the *y* direction

Fig. 7 Localization factors of a two-dimensional system: $\mu_{kx} = 1$, $\mu_{ky} = 5$, $\delta_{kx} = \delta_{ky} = 0.1$, $k_j^x = 0.9709$, $k_{j+1}^y = 4.2933$, and $k^h = k^v = k^d = k$; ---, $M = 10^3$; ——, $M = 10^4$; and ● transfer matrix method $M = 10^7$.

bending stiffnesses of the $(M+1, M+1)$th cantilever are randomly generated as $k_j^x = 0.9709$, $k_{j+1}^y = 4.2933$.

First-order approximate localization factors for a vibration mode in which vibration is originated in the x direction are shown in Fig. 7a for different values of spring stiffnesses. It is seen that the localization factor in the x direction ($\theta = 90$ and 270 deg) is smaller than that in the y direction ($\theta = 0$ and 180 deg).

First-order approximate localization factors for a vibration mode in which vibration is originated in the y direction are shown in Fig. 7b for different values of spring stiffnesses. It is seen that the localization factor in the y direction ($\theta = 0$ and 180 deg) is smaller than that in the x direction ($\theta = 90$ and 270 deg).

From Fig. 7, it may be concluded that the direction in which vibration is originated corresponds to the smallest localization factor.

It is also seen from Fig. 7 that, for the given parameters, the localization factors of the vibration mode in which vibration is originated in the y direction are larger than those of the vibration mode in which vibration is originated in the x direction. This is because the bending stiffnesses of the cantilevers in the y direction are larger than those in the x direction, which results in a smaller ratio of the spring stiffness and the bending stiffness of the cantilevers or a weaker coupling and leads to a stronger localization.

In the diagonal directions, i.e., $\theta = 45$, 135, 225, and 315 deg, the localization factors may be determined using Eqs. (26) and (27) and the numerical algorithm involving transfer matrices of dimension 2×2, as presented in Ref. 2. Numerical results are shown in Figs. 6 and 7 as solid dots for $M = 10^7$. It is seen that the diagonal directions correspond to the largest localization factor.

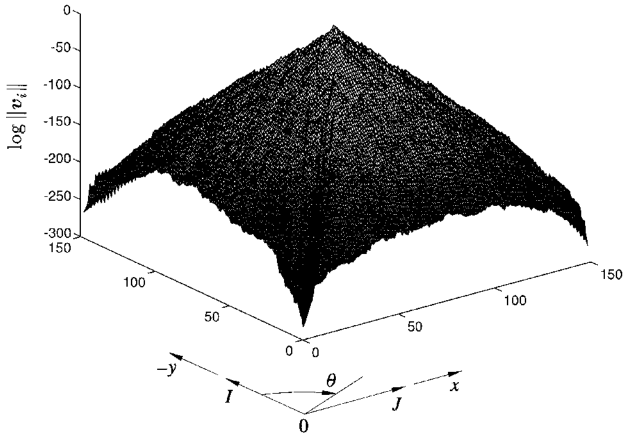
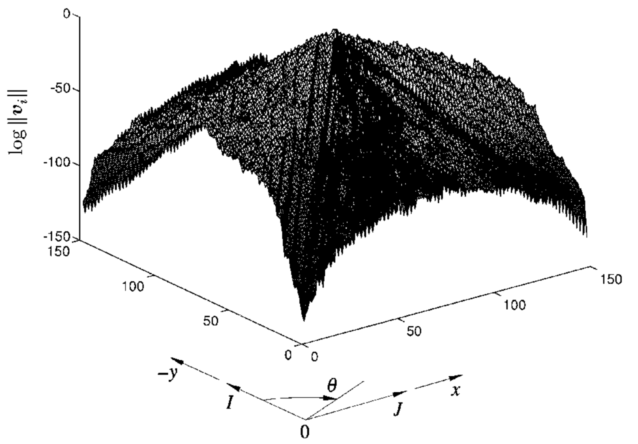
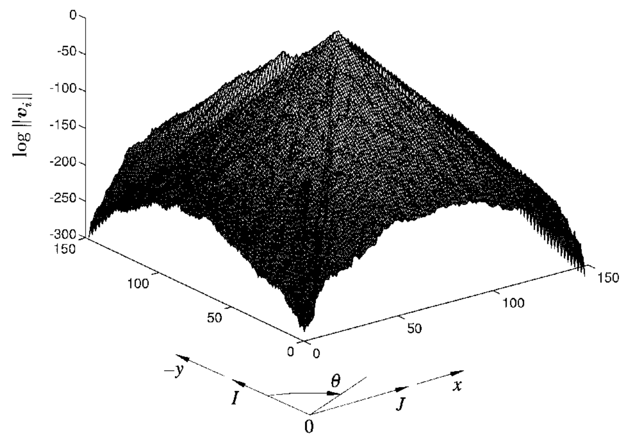


Fig. 8 Amplitudes of vibration of a two-dimensional system, vibration originated in the x direction: $\mu_{kx} = \mu_{ky} = 1$, $\delta_{kx} = \delta_{ky} = 0.1$, $k_j^x = 0.9709$, $k_{j+1}^y = 0.8587$, and $k^h = k^v = k^d = k = 0.001$.



Vibration originated in the x direction



Vibration originated in the y direction

Fig. 9 Amplitudes of vibration of a two-dimensional system: $\mu_{kx} = 1$, $\mu_{ky} = 5$, $\delta_{kx} = \delta_{ky} = 0.1$, $k_j^x = 0.9709$, $k_{j+1}^y = 4.2933$, and $k^h = k^v = k^d = k = 0.01$.

Typical vibration modes are plotted in Figs. 8 and 9 in logarithmic scale for different values of the spring stiffnesses and different directions in which vibration is originated. In Figs. 8 and 9, only the amplitudes of vibration of the cantilevers on the Mth layer in the Mth-order perturbation are plotted; therefore, the results are approximate. The cantilever–spring array has 151 rows and 151 columns; the cantilever at which vibration is originated is located at (76, 76), and the total number of perturbation terms is 75. As expected, the vibration modes of a two-dimensional system are, in general, of a hill shape when plotted in the logarithmic scale; amplitudes of vibration of the cantilevers decay linearly in the logarithmic scale away from the cantilever at which vibration is originated. The direction in which vibration is originated corresponds to the smallest rate of decay of amplitudes of vibration, whereas the diagonal directions ($\theta = 45$, 135, 225, and 315 deg) correspond to the largest rate of decay of amplitudes of vibration.

IV. Conclusions

The method of regular perturbation presented in Ref. 1 was applied to study vibration mode localization in randomly disordered weakly coupled two-dimensional cantilever–spring arrays. Iterative equations were obtained to express the displacement vector of the cantilevers on the Mth layer in the Mth-order perturbation in terms of those on the $(M-1)$th layer in the $(M-1)$th-order perturbation. Localization factors, which characterize the average exponential rates of decay or growth of the amplitudes of vibration, were defined in terms of the angles of orientation. First-order approximate results of the localization factors were obtained using a combined analytical–numerical approach. For the vibration mode in which vibration is originated at the (I_0, J_0)th cantilever, the localization factors are symmetric about the axes $I = I_0$ and $J = J_0$ and the point (I_0, J_0). It was found that, for the systems under consideration, the direction in which vibration was originated corresponded to the smallest localization factor, whereas the diagonal directions ($\theta = 45$, 135, 225, and 315 deg) corresponded to the largest localization factor. When plotted in the logarithmic scale, the vibration modes were of a hill shape with the amplitudes of vibration decaying linearly away from the cantilever at which vibration was originated.

Vibration mode localization in two-dimensional cantilever–spring arrays was investigated. The formulation presented can be readily extended to other weakly coupled higher dimensional systems such as three-dimensional mass-spring arrays.

Appendix: Equations of Motion of Two-Dimensional Cantilever–Spring Arrays

Consider the free vibration of a two-dimensional cantilever–spring array as shown in Fig. 1. The free end of each cantilever is connected to its eight neighboring cantilevers by linear springs. The mass of each cantilever is lumped at the free end. The parameters and displacements of the cantilever–spring array are shown in Fig. A1.

Considering only small deformations, the extensions of the horizontal springs $K^h_{I,J-1}$, $K^h_{I,J}$ and vertical springs $K^v_{I-1,J}$, $K^v_{I,J}$ are, respectively,

$$\begin{aligned} \Delta^h_{I,J-1} &\approx x_{I,J} - x_{I,J-1} \qquad & \Delta^h_{I,J} &\approx x_{I,J+1} - x_{I,J} \\ \Delta^v_{I-1,J} &\approx y_{I-1,J} - y_{I,J} \qquad & \Delta^v_{I,J} &\approx y_{I+1,J} - y_{I,J} \end{aligned} \tag{A1}$$

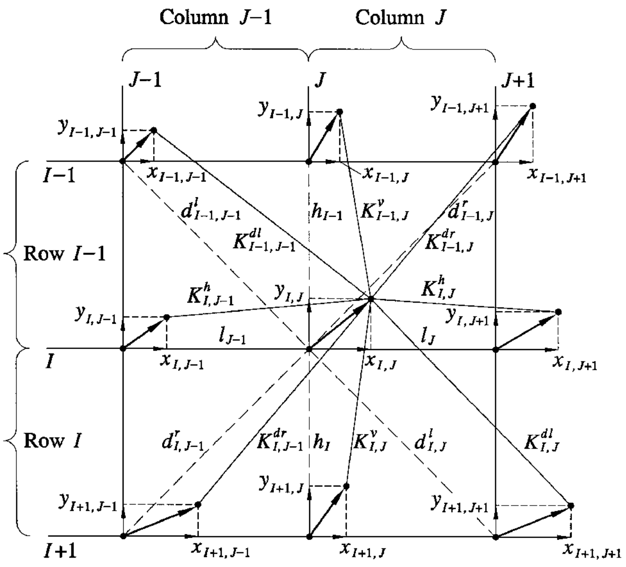


Fig. A1 Parameters and displacements of a two-dimensional cantilever–spring array.

and the extensions of the diagonal springs $K^{dl}_{I-1,J-1}$, $K^{dr}_{I-1,J}$, $K^{dr}_{I,J-1}$, and $K^{dl}_{I,J}$ are, respectively,

$$
\begin{aligned}
\Delta^{dl}_{I-1,J-1} &\approx \frac{l_{J-1}}{d^{l}_{I-1,J-1}}(x_{I,J}-x_{I-1,J-1}) + \frac{h_{I-1}}{d^{l}_{I-1,J-1}}(y_{I-1,J-1}-y_{I,J})\\
\Delta^{dr}_{I-1,J} &\approx \frac{l_{J}}{d^{r}_{I-1,J}}(x_{I-1,J+1}-x_{I,J}) + \frac{h_{I-1}}{d^{r}_{I-1,J}}(y_{I,J}-y_{I-1,J+1})\\
\Delta^{dr}_{I,J-1} &\approx \frac{l_{J-1}}{d^{r}_{I,J-1}}(x_{I,J}-x_{I+1,J-1}) + \frac{h_{I}}{d^{r}_{I,J-1}}(y_{I+1,J-1}-y_{I,J})\\
\Delta^{dl}_{I,J} &\approx \frac{l_{J}}{d^{l}_{I,J}}(x_{I+1,J+1}-x_{I,J}) + \frac{h_{I}}{d^{l}_{I,J}}(y_{I,J}-y_{I+1,J+1})
\end{aligned}
\tag{A2}
$$

The total kinetic energy of the array is

$$
T = \frac{1}{2}\sum^{N_V}\sum^{N_H} m_{I,J}\left(\dot{x}^2_{I,J} + \dot{y}^2_{I,J}\right) \tag{A3}
$$

and the total potential energy of the array is

$$
\begin{aligned}
V = \frac{1}{2}\sum^{N_V}\sum^{N_H}\Big[& K^{x}_{I,J}x^2_{I,J} + K^{y}_{I,J}y^2_{I,J} + K^{h}_{I,J-1}\left(\Delta^{h}_{I,J-1}\right)^2\\
&+ K^{h}_{I,J}\left(\Delta^{h}_{I,J}\right)^2 + K^{v}_{I-1,J}\left(\Delta^{v}_{I-1,J}\right)^2 + K^{v}_{I,J}\left(\Delta^{v}_{I,J}\right)^2\\
&+ K^{dl}_{I-1,J-1}\left(\Delta^{dl}_{I-1,J-1}\right)^2 + K^{dr}_{I-1,J}\left(\Delta^{dr}_{I-1,J}\right)^2\\
&+ K^{dr}_{I,J-1}\left(\Delta^{dr}_{I,J-1}\right)^2 + K^{dl}_{I,J}\left(\Delta^{dl}_{I,J}\right)^2\Big]
\end{aligned}
\tag{A4}
$$

Lagrange's equations of motion are of the form

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}_{I,J}}\right) - \frac{\partial T}{\partial x_{I,J}} + \frac{\partial V}{\partial x_{I,J}} &= 0\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{y}_{I,J}}\right) - \frac{\partial T}{\partial y_{I,J}} + \frac{\partial V}{\partial y_{I,J}} &= 0
\end{aligned}
\tag{A5}
$$

for $I = 1, 2, \ldots, N_V$ and $J = 1, 2, \ldots, N_H$.

Let the displacement vector be

$$
\boldsymbol{x} = \left\{x_{1,1}, y_{1,1}; x_{1,2}, y_{1,2}; \ldots; x_{N_V N_H}, y_{N_V N_H}\right\}^T
$$

Substituting Eqs. (A1–A4) into Eqs. (A5) results in the equation of motion of free vibration of the cantilever–spring array

$$
\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0} \tag{A6}
$$

Letting $\boldsymbol{x} = L\hat{\boldsymbol{x}}e^{i\omega t}$ and substituting into Eq. (A6) leads to

$$
(\boldsymbol{K} - \omega^2\boldsymbol{M})\hat{\boldsymbol{x}} = \boldsymbol{0} \tag{A7}
$$

or

$$
(\bar{\boldsymbol{A}} - \bar{\lambda}\boldsymbol{I})\hat{\boldsymbol{x}} = \boldsymbol{0} \tag{A8}
$$

where $\bar{\boldsymbol{A}} = \omega_0^{-2}\boldsymbol{M}^{-1}\boldsymbol{K}$.

Transform the nodal coordinates (I, J) to the global coordinates j, as shown in Fig. 2,

$$
j = \begin{cases} 2(I-1)N_H + 2J - 1, & j \text{ odd, for } x \text{ direction}\\ 2(I-1)N_H + 2J, & j \text{ even, for } y \text{ direction}\end{cases} \tag{A9}
$$

or

$$
I = \mathrm{int}[(j-1)/2N_H] + 1, \qquad J = \mathrm{int}[(j+1)/2] - (I-1)N_H \tag{A9$'$}
$$

Without loss of generality, it may be assumed that j is an odd number, and it corresponds to the x direction; hence, $j+1$ is an even number, and it corresponds to the y direction. Letting $u_j = \hat{x}_{I,J}$, $u_{j+1} = \hat{y}_{I,J}$, j odd, the displacement vector $\hat{\boldsymbol{x}}$ becomes $\bar{\boldsymbol{u}}$, where

$$
\bar{\boldsymbol{u}} = \left\{u_1, u_2; u_3, u_4; \ldots; u_{2N_H N_V - 1}, u_{2N_H N_V}\right\}^T \tag{A10}
$$

For simplicity of presentation, assume that there is no disorder in the geometry of the array, the lumped masses at the tips of the cantilevers, and the springs connecting the cantilevers, i.e.,

$$
\begin{gathered}
l_1 = l_2 = \cdots = l_{N_H} = h_1 = h_2 = \cdots = h_{N_V} = l\\
\hat{m}_{I,J} = 1, \qquad \hat{K}^{h}_{I,J} = k^h, \qquad \hat{K}^{v}_{I,J} = k^v, \qquad \hat{K}^{dl}_{I,J} = \hat{K}^{dr}_{I,J} = 2k^d
\end{gathered}
\tag{A11}
$$

for all I and J. The only sources of disorder are the bending stiffnesses of the cantilevers. Because K^x is the average value of the bending stiffnesses $K^x_{I,J}$ of the cantilevers (I, J) in the x direction and $\hat{K}^x_{I,J} = K^x_{I,J}/K^x$, then the expected value of $\hat{K}^x_{I,J}$ is 1.

Therefore, the nonzero elements in the jth row of matrix $\bar{\boldsymbol{A}}$ are

$$
\begin{aligned}
&\bar{A}_{j,j} = k_j \qquad \bar{A}_{j,j-2} = \bar{A}_{j,j+2} = -k^h\\
&\bar{A}_{j,j-2N_H-2} = -\bar{A}_{j,j-2N_H-1} = \bar{A}_{j,j-2N_H+2} = \bar{A}_{j,j-2N_H+3}\\
&\quad = \bar{A}_{j,j+2N_H-2} = \bar{A}_{j,j+2N_H-1} = \bar{A}_{j,j+2N_H+2}\\
&\quad = -\bar{A}_{j,j+2N_H+3} = -k^d
\end{aligned}
\tag{A12}
$$

and the nonzero elements in the $(j+1)$th row are

$$
\begin{aligned}
&\bar{A}_{j+1,j+1} = k_{j+1} \qquad \bar{A}_{j+1,j-2N_H+1} = \bar{A}_{j+1,j+2N_H+1} = -k^v\\
&-\bar{A}_{j+1,j-2N_H-2} = \bar{A}_{j+1,j-2N_H-1} = \bar{A}_{j+1,j-2N_H+2}\\
&\quad = \bar{A}_{j+1,j-2N_H+3} = \bar{A}_{j+1,j+2N_H-2} = \bar{A}_{j+1,j+2N_H-1}\\
&\quad = -\bar{A}_{j+1,j+2N_H+2} = \bar{A}_{j+1,j+2N_H+3} = -k^d
\end{aligned}
\tag{A13}
$$

where

$$
k_j = k^x_j + 2k^h + 4k^d, \qquad k_{j+1} = k^y_{j+1} + 2k^v + 4k^d \tag{A14}
$$

Acknowledgments

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